Flag-transitive linear spaces and spreads in $\mathrm{PG}(5, q)$

Finite Geometry and Friends 2023

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Introduction

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A *flag* of L is an incident point-line pair (x, ℓ).

Question: For which linear spaces L does $\mathrm{Aut}(L)$ act transitively on flags?

Linear spaces

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Linear spaces

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Due to work by Buekenhout, Delandtsheer, Doyen et al. (1990), Liebeck (1998), Saxl (2002) and others, the result is known for all L and Aut(L) except when L is constructed from a t-spread of $V(n, q)$ and Aut(L) is $T \circ G_0$, where G_0 is a subgroup of $\Gamma\mathrm{L}(1, q^n) \leq \Gamma\mathrm{L}(n, q)$.

Pauley and Bamberg (2007) studied the case $t = 2$ and $G_0 = C := \langle \omega^{q+1} \rangle \leq \Gamma\mathrm{L}(1, q^{2m})$, where ω is a generator of $\mathbb{F}_{q^{2m}}^{\times}$.

Line-spreads

Let $P(x)$ be an irreducible polynomial over $\mathbb{F}_{q^2}$ of degree m. Then $P(x)$ satisfies $\star$ if and only if for all nonzero $x, y \in \mathbb{F}_{q^2}$ we have

$$\frac{x^m P(x^{q-1})}{y^m P(y^{q-1})} \in \mathbb{F}_q \implies \frac{x}{y} \in \mathbb{F}_q.$$

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Theorem (Pauley-Bamberg, 2007)

Polynomials satisfying $\star$ $\longleftrightarrow$ Flag-transitive linear spaces with the given aut. group

Known examples

- Kantor (1993): $P(x) = x^m - \zeta$, where ζ is a generator of $\mathbb{F}_{q^2}^\times$.
- Pauley and Bamberg (2007): $\mathrm{PB}(x) = \frac{x^{p+1}-1}{x-1} - 2$ where p is an odd prime.
- Feng and Lu (2021):

$$\mathrm{FL}_n(x) := \frac{(\delta x - 1)^n - \delta(x-\delta)^n}{\delta^n - \delta}$$

where $d > 1$ is an odd divisor of $q+1$, u is a proper divisor of d, $t \in \mathbb{N}^+$, $n = d^t u$ and $\delta \in \mathbb{F}_{q^2}^\times$ is an element of order $q+1$.

Permutation polynomials (I)

Theorem (Feng-Lu, 2021)

Suppose $(\deg(P), q-1) = 1$. Then $P(x)$ satisfies $\star$ if and only if $x^d P(x^{q-1})$ permutes $\mathbb{F}_{q^2}$.

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Note that:

- Reducible $P(x)$ is interesting for permutation polynomials, but not for linear spaces.
- The case $(\deg(P), q-1) > 1$ is interesting for linear spaces, but not for permutation polynomials (since in that case $x^d P(x^{q-1})$ can never permute $\mathbb{F}_{q^2}$).

An equivalent critieron

Let $P(x) = \sum_{i=0}^{m} a_i x^i \in \mathbb{F}_{q^2}[x]$, and define $\tilde{P}(x) := \sum_{i=0}^{m} a_{m-i}^{q} x^i$.

We define a polynomial in two variables as follows.

$$H_P(z,w) := \frac{P(z)\tilde{P}(w) - \tilde{P}(z)P(w)}{z-w}.$$

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Lemma

A polynomial $P(x)$ satisfies $\star \iff$ the system $H_P(z, w) = 0$, $z^{q+1} = w^{q+1} = 1$ has no solutions with $z \neq w$.

Binomials and quadratics

Theorem (binomials)

The polynomial $P(x) = x^m - \theta$ is irreducible in $\mathbb{F}_{q^2}[x]$ and satisfies $\star$ if and only if the following hold:

(i) $\theta^{q+1} \neq 1$;

(ii) every prime factor of m divides $o(\theta)$ but not $\frac{q^2-1}{o(\theta)}$;

(iii) if $m \equiv 0 \mod 4$ then $q \equiv 1 \mod 4$;

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In particular, if $m = 3$ then there exists an irreducible cubic binomial satisfying $\star$ if and only if $q \equiv 1 \mod 3$.

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Theorem (quadratics)

There are no quadratics satisfying $\star$.

Cubics

Example ($m = 3$)

Let $P(x) = x^3 - \delta x^2 - (\delta + 3)x - 1$. Then

$$H_P(z, w) = (zw + z + 1)(zw + w + 1).$$

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$$\begin{aligned} & z^{q+1} = 1 = w^{q+1} \\ \iff & w^{q+1} + w^q + w = 0 \\ \iff & w^2 + w + 1 = 0 \\ \iff & z = w. \end{aligned}$$

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Hence $P(x)$ satisfies $\star$.

Cubics

Theorem

Let $P(x) = x^3 - \delta x^2 - \gamma x - \theta \in \mathbb{F}_{q^2}[x]$. Then $H_P(z, w)$ is reducible (and not identically zero) if and only if one of the following holds:

(i) $P(x) = B_\theta(x) := x^3 - \theta$, $\theta^{q+1} \neq 1$;

(ii) $P(x) = P_{\delta,\alpha}(x) := x^3 - \delta x^2 - (\delta\alpha + 3\alpha^{1-q})x - (\delta\alpha^2 \left(\frac{1-\alpha^{-(q+1)}}{3}\right) + \alpha^{2-q})$, $\alpha \neq 0$;

(iii) $P(x) = Q_{\delta,\gamma}(x) := x^3 - \delta x^2 - \gamma x + \delta\gamma/9$, $\gamma^{q+1} = 9$.

Furthermore

- an irreducible $B_\theta(x)$ satisfies $\star$ if and only if $q \equiv 1 \mod 3$;
- an irreducible $P_{\delta,\alpha}(x)$ satisfies $\star$ if and only if $\frac{4-\alpha^{q+1}}{3\alpha^{q+1}}$ is a nonzero square in $\mathbb{F}_q$, and $\delta = 0$ or $(\alpha + 3\delta^{-q})^{q+1} \neq 1$;
- an irreducible $Q_{\delta,\gamma}(x)$ satisfies $\star$ if and only if $\gamma^{\frac{q+1}{2}} = 3$.

Permutation polynomials (II)

In their work on characterising permutation polynomials of $\mathbb{F}_{q^2}$ of the form

$$f_{a,b}(X) = X(1 + aX^{q(q-1)} + bX^{2(q-1)}),$$

Bartoli and Timpanella (2021) considered a curve with affine equation

$$-b^{q+1} H_P(z, w) = 0$$

where $P(x) = x^3 + b^{-1}x + ab^{-1}$. They showed that $f_{a,b}(X)$ is a PP if and only if P satisfies $\star$. It follows that $P(x)$ is of the form $P_{\delta,\alpha}(x)$ with $\delta = 0$, $a = \alpha/3$ and $b = -\alpha^{q-1}/3$.

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Following their approach, we showed that for $q \geq 47$, if P satisfies $\star$ then H_P is reducible.

Equivalence

Theorem (Pauley-Bamberg, 2007)

Let $P(x), Q(x) \in \mathbb{F}_{q^2}[x]$ satisfy $\star$. Then P and Q yield equivalent linear spaces if and only if

$$P(x) = \lambda(u + v^q x)^m Q^\sigma \left(\frac{v + u^q x}{u + v^q x} \right)$$

for some $\sigma \in \mathrm{Aut}(\mathbb{F}_{q^{2m}})$ and $u, v, \lambda \in \mathbb{F}_{q^2}$ where $\lambda \neq 0$ and $u^{q+1} \neq v^{q+1}$.

Cubic equivalence

Theorem

Let $P(x)$ be an irreducible polynomial of the form $B_\theta(x)$, $P_{\delta,\alpha}(x)$ or $Q_{\delta,\gamma}(x)$ that satisfies $\star$. Then $P(x)$ is equivalent to some $P_{\delta',1}(x)$.

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By counting the number of irreducibles of the form $P_{\delta,1}(x)$, and calculating precisely the equivalences between polynomials of this form, we get the following.

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Theorem

The number of equivalence classes of irreducible cubic polynomials satisfying $\star$ such that $H_P(z,w)$ is reducible is precisely

$$\begin{cases} \frac{q-1}{3}, & \text{if } q \equiv 1 \mod 3 \\ \frac{q+1}{3}, & \text{if } q \not\equiv 1 \mod 3 \end{cases}.$$

A surprising connection

Lemma

$P(x)$ divides $H_P(x^{q^2}, x)$.

For each of our cubic orbit representatives $P_{\delta,1}(x)$ we have

$$H_{P_{\delta,1}}(z, w) = (zw + z + 1)(zw + w + 1),$$

and so $P_{\delta,1}(x)$ divides $(x^{q^2+1} + x^{q^2} + 1)(x^{q^2+1} + x + 1)$.

Conversely, methods of Stichtenoth-Topuzoğlu (2012) and Gow-McGuire (2021) tells us that every irreducible cubic factor of $(x^{q^2+1} + x^{q^2} + 1)(x^{q^2+1} + x + 1) \in \mathbb{F}_{q^2}[x]$ is of the form

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We aim to explain and expand this connection to find polynomials of other degrees satisfying $\star$.

Orbit polynomials

Lemma

$P(x)$ divides $H_P(x^{q^2}, x)$.

For $\Psi = \begin{pmatrix} -b & -d \\ c & a \end{pmatrix} \in \mathrm{GL}(2, q^2)$, let

$$H_\Psi(z, w) = czw + az + bw + d.$$

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We consider the case in which $H_P(z, w) = \prod_\Psi H_\Psi(z, w)$.

There has much study of the factorisation of

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If $P(x) \mid F_\Psi(x)$ and $Q(x) \mid F_\Phi(x)$ then P and Q are equivalent if and only if F_Ψ and F_Φ are equivalent.

Orbit polynomials

$$F_\Psi(x) := H_\Psi(x^{q^2}, x) = cx^{q^2+1} + ax^{q^2} + bx + d$$

The number of factors of F_Ψ and their degrees were determined by Stichtenoth and Topuzoğlu (2011).
Further work was carried out by Gow and McGuire (2022) using results on group actions and orbit polynomials.

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For $\Psi \in \mathrm{GL}(2, q^2)$, let $[\Psi]$ denote the corresponding element of $\mathrm{PGL}(2, q^2)$.
For $s = [\Psi] = \left[\begin{pmatrix} -b & -d \\ c & a \end{pmatrix}\right]$, define $s(x) = -\frac{bx+d}{cx+a}$.

The *orbit polynomial* of the group G generated by s is

$$O_G(x) = \prod_{s \in G} (x - s(y)) \in \mathbb{F}_{q^2}(y)[x].$$

Orbit polynomials

$$F_\Psi(x) := H_\Psi(x^{q^2}, x) = cx^{q^2+1} + ax^{q^2} + bx + d$$

The *orbit polynomial* of the group G generated by $s = [\Psi]$ is

$$O_G(x) = \prod_{s \in G} (x - s(y)) \in \mathbb{F}_{q^2}(y)[x].$$

Gow-McGuire (2022)

Let $|G| =: r$ divide $q^2 + 1$. Then the irreducible factors of $F_\Psi(x)$ have degree r and each irreducible factor is a specialisation of the orbit polynomial $O_G(x)$.

Orbit polynomials

Example

Let $\Psi = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, so $F_\Psi(x) = x^{q^2+1} + x + 1$ and the order of $s = [\Psi]$ is 3.

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$$\begin{aligned} O_G(x) &= (x-y)(x-s(y))(x-s^2(y)) \\ &= (x-y)\left(x + \frac{y+1}{y}\right)\left(x + \frac{1}{y+1}\right) \\ &= x^3 + \left(\frac{1+3y-y^3}{y(y+1)}\right)x^2 + \left(\frac{1-3y^2-y^3}{y(y+1)}\right)x - 1 \\ &= P_{\delta,1}(x), \end{aligned}$$

where $\delta = \frac{1+3y-y^3}{y(y+1)}$.

Remarks

Pauley-Bamberg

Let $\mathrm{PB}(x) = \frac{x^{p+1}-1}{x-1} - 2$, where p is an odd prime.

Let

$$M = \left\{ \begin{pmatrix} (1+i)/i & -1 \\ 1 & (1-i)/i \end{pmatrix} : i \in \mathbb{F}_p^* \right\}.$$

Then the elements of M have order p and $\mathrm{PB}(x)$ is a factor of $F_\Psi(x)$ for some $\Psi \in M$.

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Then the elements of M have order p and $\mathrm{PB}(x)$ is a factor of $F_\Psi(x)$ for some $\Psi \in M$.

Furthermore, all polynomials that are specialisations of an orbit polynomial O_G, where $|G| = p$, are equivalent.

Remarks

Feng-Lu

Feng and Lu (2021) showed that

$$\mathsf{FL}_n(x) := \frac{(\delta x - 1)^n - \delta(x-\delta)^n}{\delta^n - \delta}$$

satisfies $\star$, where $d > 1$ is an odd divisor of $q+1$, u is a proper divisor of d, $t \in \mathbb{N}^+$, $n = d^t u$ and $\delta \in \mathbb{F}_{q^2}^{\times}$ is an element of order $q+1$.

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We have

$$\mathsf{FL}_3(x) = P_{0,-(\delta+\delta^{-1})}(x) \in \mathbb{F}_q[x].$$

Not every irreducible cubic satisfying $\star$ is equivalent to one of the form $\mathsf{FL}_3(x)$, and so this construction is a proper subset of ours for the case $m = 3$.

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Let $C_n = \left\{ \begin{pmatrix} \lambda\delta^q - \delta & 1-\lambda \\ \lambda - 1 & \delta^q - \lambda\delta \end{pmatrix} : \lambda^n = 1, \right\}$. Then

$$H_{\mathrm{FL}_n}(z,w) = \prod_{\Psi \in C_n \setminus \{I\}} H_\Psi(z,w)$$

and

$$\mathrm{FL_n}(x) = \prod_{\Psi \in C_n} (x - [\Psi](y))$$

for some y.

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for some y. **For different choices of y we obtain new examples inequivalent to $\mathrm{FL}_n(x)$.**

Quartics

Theorem

If $P(x)$ is an irreducible quartic satisfying $\star$, then H_P is absolutely irreducible.

From Magma computation:

- there **are** irreducible quartic polynomials satisfying $\star$ for every $q < 9$. For each of these polynomials P, H_P is absolutely irreducible (which never occurs in the cubic case).
- there **are no** irreducible quartic polynomials satisfying $\star$ for $9 \leq q \leq 23$.

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- there **are no** irreducible quartic polynomials satisfying $\star$ for $9 \leq q \leq 23$.

Following the method of Bartoli-Timpanella in using the Aubry-Perret bound on curves, we can conclude there are no quartics satisfying $\star$ for $q > 409$.

Summary

In summary:

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- We can use this to extend the known families to new inequivalent examples of the same degree.

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Preprint: *Cyclic 2-Spreads in* $V(6, q)$ *and Flag-Transitive Affine Linear Spaces* (arXiv:2309.06872)

Thank you for your attention!