Non-linear MRD codes

from cones over exterior sets

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Rank Distance Codes

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$\mathcal{C}' \subseteq \mathbb{F}_q^{(m-u)\times u}$ obtained by $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ deleting the last u rows of $\mathcal{C}$,

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Proposition. [E. Byrne, A. Ravagnani (2017)]

Let $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ be an MRD code. Then, for any $1 \le u \le m-1$, the punctured code $\mathcal{C}'$ is MRD as well.

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$$\mathcal{C}_2 = \{ f \circ \alpha^\rho \circ g + h \ \mid \alpha \in \mathcal{C}_1 \}$$

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- (Generalized) Delsarte-Gabidulin code
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$$\mathcal{G}_{k,\sigma} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n} \right\},$$

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- (Generalized) twisted Gabidulin code
  [J. Sheekey (2016)- G. Lunardon, R. Trombetti, Y. Zhou (2018)]

$$\mathcal{H}_{k,\sigma}(\eta, h) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \eta \alpha_0^{q^h} x^{\sigma^k} : \alpha_i \in \mathbb{F}_{q^n} \right\},$$

  $0 < k < n, 0 \leq h < n, \ \eta \in \mathbb{F}_{q^n}$ s.t. $\mathrm{N}_{q^{sn}/q^s}(\eta) \neq (-1)^{nk}$.

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- Trombetti-Zhou code
  [R. Trombetti, Y. Zhou (2019)]

$$\mathcal{D}_{k,\sigma}(\xi) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \xi \alpha_k x^{\sigma^k} : \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{F}_{q^n}, \alpha_0, \alpha_k \in \mathbb{F}_{q^t} \right\}$$

  $n = 2t, 2 \leq d \leq n$ and q odd , $\xi \in \mathbb{F}_{q^n}$ such that $\mathrm{N}_{q^n/q^t}(\xi) \notin \square$.

Known MRD codes

Additive $(n, n, n-k+1)$-MRD codes

- [K. Otal, F. Özbudak (2017)]

$$\mathcal{A}_{k,\sigma,q_0}(\eta, h) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \eta \alpha_0^{q_0^h} x^{\sigma^k} : \alpha_i \in \mathbb{F}_{q^n} \right\},$$

$n, k, u, h > 0$ and $k < n$, $q = q_0^u$, $\eta \in \mathbb{F}_{q^n}$ s.t. $\mathrm{N}_{q^{sn}/q_0^s}(\eta) \neq (-1)^{nku}$ ($\mathbb{F}_{q_0}$-linear code).

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Non-linear $(n, n; q, n-k+1)$-MRD codes

- [K. Otal, F. Özbudak (2018)]

$$\mathcal{C}_{n,k,\sigma,l} = \mathcal{C}^{(1)}_{n,k,\sigma,l} \cup \mathcal{C}^{(2)}_{n,k,\sigma,l}$$

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$1 \leq k \leq n-1$ and $I \subseteq \mathbb{F}_q$.

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- [G. Donati, N. Durante (2018)]: $n \geq 3, 1 \leq d \leq n-1$, $(d+1, n; q, d)$-MRD codes.

Field Reduction

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Let ξ be a root of an irreducible polynomial of degree n over $\mathbb{F}_q$. Then, for any $x \in \mathbb{F}_{q^n}$ one has

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$$\Phi : (x_1, \ldots, x_m) \in \mathbb{F}_{q^n}^m \longmapsto \begin{pmatrix} x_{10} & x_{11} & \ldots & x_{1,n-1} \\ x_{20} & x_{21} & \ldots & x_{2,n-1} \\ x_{m0} & x_{m1} & \ldots & x_{m,n-1} \end{pmatrix} \in \mathbb{F}_q^{m\times n},$$

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1. Φ maps h-dim. subspaces of $\mathbb{F}_{q^n}^m$ to hn-dim. subspaces of $\mathbb{F}_q^{mn}$.

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1. Φ maps h-dim. subspaces of $\mathbb{F}_{q^n}^m$ to hn-dim. subspaces of $\mathbb{F}_q^{mn}$.
2. Φ induces a map

$$\Phi' : \mathrm{PG}(m-1, q^n) \longrightarrow \mathrm{PG}(\mathbb{F}_q^{m\times n}, \mathbb{F}_q) = \mathrm{PG}(mn-1, q)$$

sending $(h-1)$-dim. proj. subspaces to $(hn-1)$-dim. proj. subspaces.

Segre Variety

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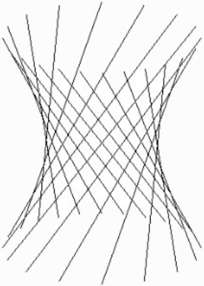
$$\mathcal{S}_{m-1,n-1} = \{\langle M\rangle_{\mathbb{F}_q} \in \mathrm{PG}(mn-1,q) \colon \mathrm{rk}\, M = 1\}$$

Segre variety of $\mathrm{PG}(mn-1,q)$, $m \leq n$



Segre Variety

$$\mathcal{S}_{m-1,n-1} = \{\langle M\rangle_{\mathbb{F}_q} \in \mathrm{PG}(mn-1,q) \colon \mathrm{rk}\, M = 1\}$$

Segre variety of $\mathrm{PG}(mn-1,q)$, $m \leq n$

$\mathcal{S}_{m-1,n-1}$ can be seen as the field reduction of the set in $\mathrm{PG}(m-1,q^n)$, $m \leq n$

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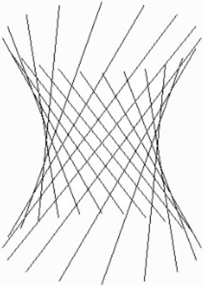

Secant variety & exterior sets

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An exterior set $\mathcal{E}$ wrt $\Omega_h(\mathcal{A})$ maximal in size is called maximum (wrt $\Omega_h(\mathcal{A})$)

Exterior sets & RD-codes

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i) the set

$$\mathcal{C} = \{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{E}', \rho \in \mathbb{F}_q\}$$

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ii) $|\mathcal{E}| \leq \frac{q^{m(n-h-1)}-1}{q-1}$

In addition, if $\mathcal{E}$ is maximum then $\mathcal{C}$ is MRD.

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THEOREM. [G. Donati, N. Durante (2018)]

Any C_F^σ-set is projectively equivalent to the set

$$\mathcal{X} = \{A, B\} \cup \bigcup_{a \in \mathbb{F}_q^*} \mathcal{X}_a,$$

with

- $A = (0, \ldots, 0, 1)$, $B = (1, 0, \ldots, 0)$ vertices of $\mathcal{X}$,
- $\mathcal{X}_a = \{(1, t, t^{\sigma+1}, \ldots, t^{\sigma^{d-1}+\ldots+\sigma+1}) : \mathrm{N}_{q^n/q}(t) = a\}$ components of $\mathcal{X}$.

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C_F^σ-SETS

- Any component $\mathcal{X}_a \cong \mathrm{PG}(n-1, q)$.
- The line AB can be partitioned in

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where

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Exterior set from a C_F^σ-set

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Theorem. [G. Donati, N. Durante (2018)]

Let

$$\mathcal{X}_1 = \{(x, x^{\sigma}, \ldots, x^{\sigma^d}) : x \in \mathbb{F}_{q^n}^*\} \cong \mathrm{PG}(n-1, q)$$

and $\Pi \cong \mathrm{PG}(d, q)$ be a subgeometry of $\mathcal{X}_1$. For any $T \subseteq \mathbb{F}_q^*, 1 \in T$, the set

$$\mathcal{E} = \left(\mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a\right) \cup \bigcup_{a \in T} J_a$$

is a maximum exterior set with respect to $\Omega_{d-2}(\Pi)$.

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Corollary. G. Donati, N. Durante (2018)]

The exterior set $\mathcal{E}$ gives rise to a $(d+1, n, q; d)$-MRD code $\mathcal{C}$, with $q > 2$, $n \geq 3$ and $2 \leq d \leq n-1$.

Embedding of Σ & Cone

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- $M,N\subseteq\mathrm{PG}(n-1,q^n)$ two disjoint sets with M a subspace,

$$\mathcal{K}(M,N)=\bigcup_{P\in M,Q\in N}PQ$$

cone with vertex M and base N

Exterior sets from cones

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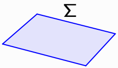

EXTERIOR SETS FROM CONES

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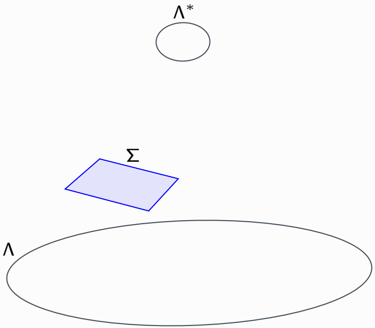

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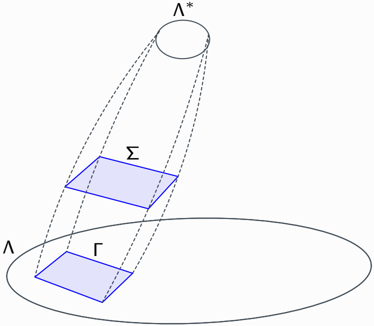

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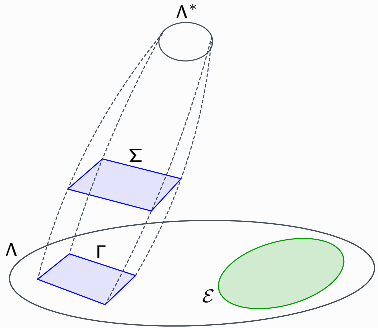

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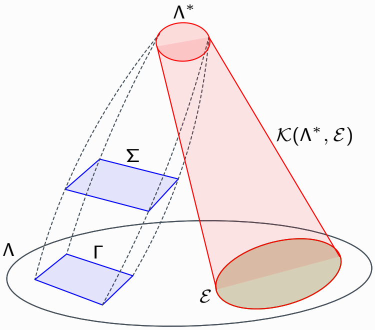
Then, $\mathcal{K} = \mathcal{K}(\Lambda^*, \mathcal{E})$ is a (maximum) exterior set with respect to $\Omega_{n-k-1}(\Sigma)$.



Our construction

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- $$J_a = \{(1, \underbrace{0, \ldots, 0}_{n-k}, (-1)^{n-k} t, 0, \ldots, 0) : \mathrm{N}_{q^n/q}(t) = a\}$$

for any $T \subseteq \mathbb{F}_q^*, 1 \in T$,

$\mathcal{E}$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Gamma)$

The code $\mathcal{C}_{\sigma,T}$

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COROLLARY [N. Durante, G.G. Grimaldi, GL (202x)]

For any $T \subseteq \mathbb{F}_q^*, 1 \in T$, the set $\mathcal{K} = \mathcal{K}(\Lambda^*, \mathcal{E})$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Sigma)$. Then, the set

$$\mathcal{C}_{\sigma,T} = \{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{K}, \rho \in \mathbb{F}_q\}$$

is an $(n, n, q; n-k+1)$-MRD code.

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- [G. Donati, N. Durante (2018)] $n \geq 3, 2 \leq d \leq n-1$, $(n, n; q, n-2) \Longrightarrow$ punctured code $\mathcal{C}'_{\sigma,T} \subseteq \mathbb{F}_q^{(n-k+2)\times n}$ obtained by $\mathcal{C}_{\sigma,T}$ deleting the last $(k-2)$ rows.

$\mathcal{C}_{\sigma,T}$ AS SET OF σ-LINEARIZED POLYNOMIALS

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$$\mathcal{C}_{\sigma,T} = \left\{ \sum_{i=0}^{d} \lambda \alpha^{\sigma^i} \xi^{\frac{\sigma^i - 1}{\sigma - 1}} x^{\sigma^i} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda, \alpha, \beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\xi) \in \mathbb{F}_q^* \setminus T \right\}$$

$$\cup \left\{ \lambda \alpha x + (-1)^{d+1} \lambda \alpha^{\sigma} \eta x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda, \alpha, \beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\eta) \in T \right\}$$

$$\cup \left\{ \alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} \cup \left\{ \alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\}$$

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$$\mathcal{U} = \left\{ \alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \,:\, \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^d} \,:\, f \in \mathcal{G}_{k-1,\sigma} \right\} \cong \mathcal{G}_{k-1,\sigma}$$

$\mathcal{C}_{\sigma,T}$ AS SET OF σ-LINEARIZED POLYNOMIALS

$$\begin{aligned}\mathcal{C}_{\sigma,T} = & \left\{ \sum_{i=0}^{d} \lambda \alpha^{\sigma^i} \xi^{\frac{\sigma^i - 1}{\sigma - 1}} x^{\sigma^i} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \, : \, \lambda, \alpha, \beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\xi) \in \mathbb{F}_q^* \setminus T \right\} \\ & \cup \left\{ \lambda \alpha x + (-1)^{d+1} \lambda \alpha^{\sigma} \eta x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \, : \, \lambda, \alpha, \beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\eta) \in T \right\} \\ & \cup \left\{ \alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \, : \, \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} \cup \left\{ \alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \, : \, \alpha, \beta_i \in \mathbb{F}_{q^n} \right\}\end{aligned}$$

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$$\mathcal{V} = \left\{ \alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} \, : \, \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^{d+1}} \, : f \in \mathcal{G}_{k-1,\sigma} \right\} \cong \mathcal{G}_{k-1,\sigma}$$

Equivalence issue

$$\mathcal{C}_{n,k,\sigma,I} = \mathcal{C}^{(1)}_{n,k,\sigma,I} \cup \mathcal{C}^{(2)}_{n,k,\sigma,I}$$

where

$$\mathcal{C}^{(1)}_{n,k,\sigma,I} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_0) \in I \right\}$$

$$\mathcal{C}^{(2)}_{n,k,\sigma,I} = \left\{ \sum_{i=1}^{k} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_k) \notin (-1)^{k(n+1)} I \right\}$$

$1 \leq k \leq n-1$ and $I \subseteq \mathbb{F}_q$.

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Theorem. [N. Durante, G.G.Grimaldi, GL (202X)]

Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, the code $\mathcal{C}_{n,k,\sigma,I}$ contains a unique subspace equivalent to $\mathcal{G}_{k-1,\sigma}$.

Equivalence issue

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Theorem. [N. Durante, G.G.Grimaldi, GL (202x)]

If $q=2$ or $T=\mathbb{F}_q^*$ and $I \in \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are both equivalent to a $\mathcal{G}_{k,\sigma}$.
If $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$ and $1 \in T \subseteq \mathbb{F}_q^*$. Then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are neither equivalent nor adjointly equivalent.

Dankuwel

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