Unveil the ghosts

Silvia Pagani (Università Cattolica del Sacro Cuore, Brescia, Italy)

Finite Geometry & Friends
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joint work with Marco Della Vedova (Chalmers) and Silvia Pianta (UniCatt)

Ghosts

Let $q = p^h$.

Definition

A (multi-)subset S of $\mathrm{PG}(n, q)$ is said to be a ghost if it has constant intersection size (modulo p) with hyperplanes.

- By multisets we mean multisets modulo p, where each point may be counted up to $p - 1$ times.
- Examples of ghosts: the empty set, k-dimensional subspaces ($k \geq 1$).
- The (multi-)complement of a ghost is a ghost.
- Operation to be considered: multiset sum modulo p.

Polynomial characterization

Let $\mathbf{X} = (X_0, \ldots, X_n)$.

The Rédei factor of a point $P = (p_0, \ldots, p_n) \in \mathrm{PG}(n, q)$ is

$$P \cdot \mathbf{X} = p_0 X_0 + \ldots p_n X_n.$$

Let $S = \{P_1, \ldots, P_s\}$ be a (multi-)subset of $\mathrm{PG}(n, q)$. The power sum polynomial G^S of S is defined as

$$G^S(\mathbf{X}) = \sum_{i=1}^{s} (P_i \cdot \mathbf{X})^{q-1}.$$

Theorem

S is a ghost if and only if $G^S \equiv 0$.

In the plane

From now on: $n = 2$.

Examples of ghosts of $\mathrm{PG}(2, q)$:

- lines;
- for q square: Baer subplanes and unitals;
- for q even: hyperovals.

Coding theory

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Theorem (Della Vedova, P., Pianta 2023)

Ghosts, endowed with the multiset sum modulo p, constitute a vector space $\mathcal{G}$ over $\mathbb{F}_p$.

Moreover $\mathcal{G} = \langle C, C^\perp \rangle = C^\perp \oplus \langle \mathbf{j} \rangle$ and
$\mathcal{G} = C^\perp \sqcup (C^\perp + \mathbf{j}) \sqcup (C^\perp + 2\mathbf{j}) \sqcup \ldots \sqcup (C^\perp + (p-1)\mathbf{j})$.

Dimensions

It results

$$\begin{aligned} \dim C &= \binom{p+1}{2}^h + 1, \\ \dim C^\perp &= p^{2h} + p^h - \binom{p+1}{2}^h, \\ \dim \mathcal{G} &= p^{2h} + p^h + 1 - \binom{p+1}{2}^h. \end{aligned}$$

If $q = p$ prime, then $\mathcal{G} = C$ and all ghosts are multi-sums of lines.

If $h > 1$, then $C^\perp \not\leq C$ and other ghosts arise.

Ghosts of PG$(2, 2)$ and PG$(2, 3)$

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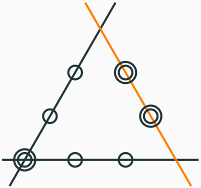
In PG$(2,3)$ points may be counted twice. There are 3^7 ghosts, including multiset sums of double lines and simple lines.



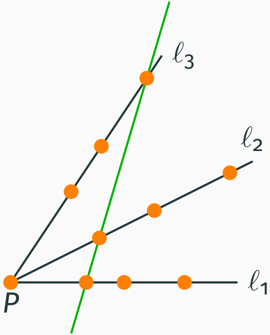
Ghosts of PG$(2, 4)$

size	ghost	size	ghost	size	ghost	size	ghost
0		21		8		13	
5		16		9		12	
6		15		10		11	
7		14					

Projective triads

Definition

Let q be even. Let ℓ_1, ℓ_2, ℓ_3 be three lines through a point P. Let $B_\iota \subset \ell_\iota \setminus \{P\}$, $|B_\iota| = \frac{q}{2}$, $\iota \in \{1, 2, 3\}$, such that a line connecting a point of B_i to a point of B_j also meets B_k, $\{i, j, k\} = \{1, 2, 3\}$.
The set $B_1 \cup B_2 \cup B_3 \cup \{P\}$ is called projective triad.



A projective triad is a minimal blocking set of $\mathrm{PG}(2, q)$ of size $\frac{3q}{2} + 1$ and a ghost.

Linear sets

Definition

Let $\mathrm{PG}(r-1, q^s)$ be defined from $V(r, q^s)$. Let $U \leq_{\mathbb{F}_q} V(r, q^s)$, $\dim U = t$. A subset L of $\mathrm{PG}(r-1, q^s)$ is an $\mathbb{F}_q$-linear set of rank t if it contains the spans (over $\mathbb{F}_{q^s}$) of non-zero vectors of U:

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^s}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

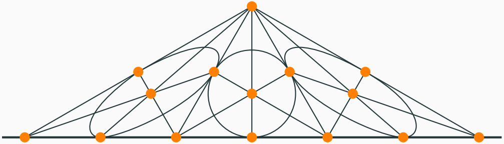
Property: for each projective subspace Λ intersecting L_U, it results $|L_U \cap \Lambda| \equiv 1 \bmod q$.

Therefore, a linear set $L \subseteq \mathrm{PG}(2, q^s)$ is a ghost $\iff$ it is a blocking set $\iff t > s$.

Linear sets

In PG$(2, 8)$ ($t > 3$):

- linear sets of rank 4: lines; projective triads; configuration of 15 points;



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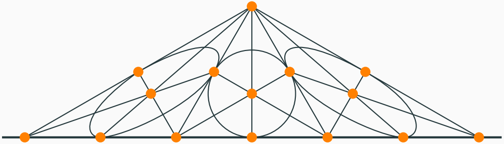
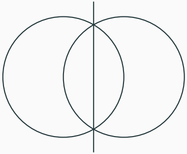


- linear sets of rank 5 include: lines; configuration of 25 points;



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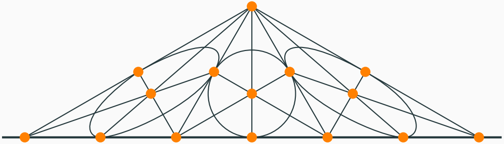

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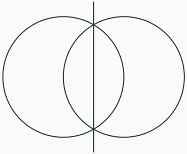

- linear sets of rank $t > 6$: PG$(2, 8)$.

Thank you for your attention!