On the Structure of Large Equidistant Grassmannian Codes

Ago-Erik Riet[1]
joint work with Daniele Bartoli, Leo Storme and Peter Vandendriessche; Jozefien D'haeseleer and Giovanni Longobardi

Finite Geometry and Friends, Brussels

June 2019

[1]agoerik@ut.ee, University of Tartu - My work was supported by COST action IC-1104: Random network coding and designs over GF(q); Ghent University; and Estonian Research Council through the research grants PUT405, IUT20-57, PSG114.

Codes, Anticodes, Erdős-Ko-Rado problem

- A code is a subset of a metric space with pairwise minimum distance $\geq d$, its elements called codewords.
- An anticode or an Erdős-Ko-Rado set/family is a subset of a metric space with pairwise maximum distance $\leq d$.
- In a graph, $V =$ codewords and $x \sim y \Leftrightarrow \text{dist}(x, y) \geq d$ they are cliques and independent sets respectively.
- There may be a natural generalization of a distance $\mathrm{d} : C^{(2)} \to \mathbb{R}$ to a function $\mathrm{d} : C^{(k)} \to \mathbb{R}$, think of span/union/intersection size or dimension for constant-sized or constant-dimension codewords.

Codes, Anticodes, Erdős-Ko-Rado problem

- T. Etzion posed the question of restricting the threewise intersection dimension in a collection of subspaces.
- Motivated by this, say an $(d_1, D_1; d_2, D_2; \ldots, d_m, D_m)$-family is a collection of subspaces in a projective space, each of dimension (non-strictly) between d_1 and D_1, the dimension of the intersection of every pair of them between d_2 and D_2 etc.
- Call this family proper if each bound d_i and D_i is attained.
- In a proper family with $D_{i-1} = d_i$ all codewords share a common d_i-space.
- In this talk I will next only talk about $(d_1 = D_1; d_2 = D_2)$-families. We also have work in progress about $(d_1 = D_1; \ldots; d_3 = D_3)$-families.

Constant intersection Grassmannian Codes

- Denote the q-element finite field by $\mathbb{F}_q$. The Grassmannian $\mathcal{G}_q(m, k)$ is the set of all k-dimensional vector subspaces of the m-dimensional vector space $\mathbb{F}_q^m$.
- A constant dimension subspace code or a Grassmannian code is a subset of $\mathcal{G}_q(m, k)$. Its elements are codewords.
- Projectively, a code $\subseteq \mathcal{G}_q(m, k)$ is a collection of (projective) $(k-1)$-spaces contained in a (projective) $(m-1)$-space $\mathbf{PG(m-1, q)}$.

Constant intersection Grassmannian Codes

- A Grassmannian code is equidistant or constant distance or constant intersection if every pair of codewords intersect in a subspace of some fixed dimension t. It is also called a t-intersecting constant dimension code.
- Then say $C \subseteq \mathcal{G}_q(m, k)$ is a $(k-1, t-1)$**-code**. Here we have projective dimension, which equals vector dimension minus 1.
- Assume dimension $m-1$ of ambient projective space $\mathrm{PG}(m-1, q)$, equivalently of $\mathbb{F}_q^m$, is sufficiently large.

$(k-1, t-1)$-codes

- A *sunflower* is a (k, t)-code such that all codewords share a common t-space. Thus they are pairwise disjoint outside this t-space. On quotienting, equivalent to a partial $(k-t-1)$-spread.
- Let $C \subseteq \mathcal{G}_q(*, k)$ be a $(k-1, t-1)$-code. Etzion and Raviv [*Equidistant codes in the Grassmannian, 2013*] notice that, via a reduction to classical binary equidistant constant weight codes and results of Deza, and, Deza and Frankl:
  If C is not a sunflower then

$$|C| \leq \left(\frac{q^k - q^t}{q-1}\right)^2 + \frac{q^k - q^t}{q-1} + 1.$$

$(k-1, t-1)$-codes

- If C is not a sunflower then

$$|C| \leq \left(\frac{q^k - q^t}{q-1}\right)^2 + \frac{q^k - q^t}{q-1} + 1.$$

- Conjecture (Deza): If C is not a sunflower then

$$|C| \leq \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q = \frac{q^{k+1} - 1}{q-1}.$$

- **Theorem** [*Bartoli, R., Storme, Vandendriessche*]. If C is not a sunflower and $t = 1$ then

$$|C| \leq \left(\frac{q^k - q}{q-1}\right)^2 + \frac{q^k - q}{q-1} + 1 - q^{k-2}.$$

$(2,0)$-codes

Beutelspacher, Eisfeld, Müller [*On Sets of Planes in Projective Spaces Intersecting Mutually in One Point, 1999*]:

- For projective planes pairwise intersecting in a projective point:
  - the set of points in ≥ 2 codewords spans a subspace of projective dimension ≤ 6;
  - there are up to isomorphism only 3 codes C where this projective dimension is 6, all related to the Fano plane.
- For $q \neq 2$ and $|C| \geq 3(q^2+q+1)$:
  - C is contained in a Klein quadric in $PG(5,q)$, or
  - is a dual partial spread in $PG(4,q)$, or
  - all codewords have a point in common.

$(2, 0)$-codes, $q = 2$

- For projective planes pairwise intersecting in a projective point, for $q = 2$:
  Deza's Conjecture: If C is not a sunflower then

$$|C| \leq 15.$$

- Bartoli and Pavese [*A note on equidistant subspace codes, 2015*] disproved it and found a code with

$$|C| = 21,$$

with a unique such example.

$(n, n-t)$-codes

- A code of projective n-spaces pairwise intersecting exactly in an $(n-t)$-space.
- An *intersection point* is a point contained in ≥ 2 codewords.
- The *base* $\mathcal{B}(S)$ of a codeword S is the span of intersection points contained in it.
- Extending the definition of a code

$$C \subseteq \mathcal{G}_q(*, n)$$

to a code

$$C \subseteq \mathcal{G}_q(*, n) \cup \mathcal{G}_q(*, n-1) \cup \ldots,$$

we may replace each codeword by its base.

Primitive $(n, n-t)$-codes

- If the ambient projective space is $(2n+1-\delta)$-dimensional, the dual of an $(n, n-t)$-code is an $(n-\delta, n-\delta-t)$-code.
- If $\exists$ a point contained in all codewords then, upon quotienting by it, we have an $(n-1, n-1-t)$-code.
- An $(\leq n, n-t)$-code is a collection of **at-most-n-spaces** pairwise intersecting exactly in an $(n-t)$-space.
- An $(n, n-t)$-code C is *primitive* (old definition by Eisfeld) if
  1. all $\mathcal{B}(S) := \langle S \cap T : T \in C \backslash \{S\} \rangle$, where $S \in C$, are n-dimensional;
  2. ambient space has dimension at least $2n+1$.
  3. there is no point contained in all codewords;
  4. ambient space is the span of all codewords;
  5. $S = \mathcal{B}(S)$ for all $S \in C$.

New primitivity

- To make primitivity definition self-dual, should add:

6. For all codewords $S \in C$: $S = \bigcap_{T \in C \setminus \{S\}} \langle S, T \rangle$.

- So, say an $(n, n-t)$-code C is *'new' primitive* (new definition by us) if 1. - 6. hold.
- Conditions 3. and 4. are dual.
  Conditions 5. and 6. are dual.
- Condition 2. allows induction on n by dualisation.
- Conditions 3. and 4. allow induction on n by quotienting.
- Definition remains self-dual if generalised to codewords of several dimensions and several intersection dimensions, i.e. if we keep 3. - 6.

$(n, n-t)$-codes with small t

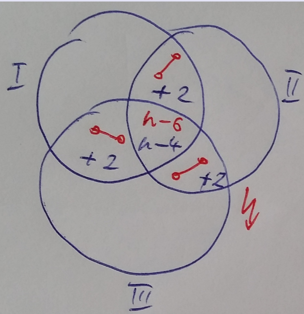
- For $t = 0$ we have $|C| = 1$.
- For $t = 1$: for an $(n, n-1)$-code, equivalently, intersections are **at least** dimension $n - 1$.
- By geometric Erdős-Ko-Rado: then all codewords
  1) share a common $(n-1)$-space, i.e. they form a *sunflower*, or,
  2) are contained in a common $(n+1)$-space (since any codeword S is contained in $\langle S_1, S_2 \rangle$ for some codewords S_1, S_2 such that $S_1 \cap S_2 \not\subseteq S$), i.e. they form a *ball*.
- Thus $(n, n-1)$-codes are classified.

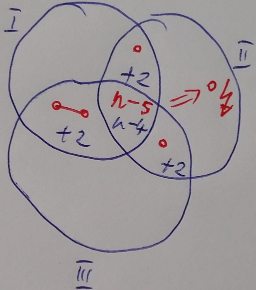
Classifying $(n, n-2)$-codes

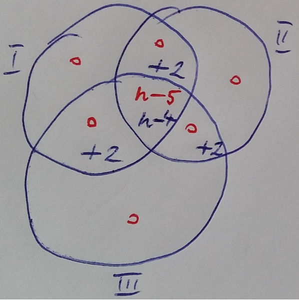
- If $\exists$ a point in common in all codewords of an $(n, n-2)$-code, quotient by it to get an $(n-1, n-3)$-code. Such codes are thus classified by induction on n.
- We may assume $\langle S : S \in C \rangle$ is the ambient space. (Intersection properties do not change; otherwise, in the dual code there is a point in common in all codewords.)
- If ambient space dimension is $2n + 1 - \delta$ then the **dual** of
  - an $(n, n-t)$-code is an $(n-\delta, n-t-\delta)$-code;
  - an $(\leq n, n-t)$-code is an $(\geq n-\delta, n-t-\delta)$-code.

Classifying $(n, n-2)$-codes

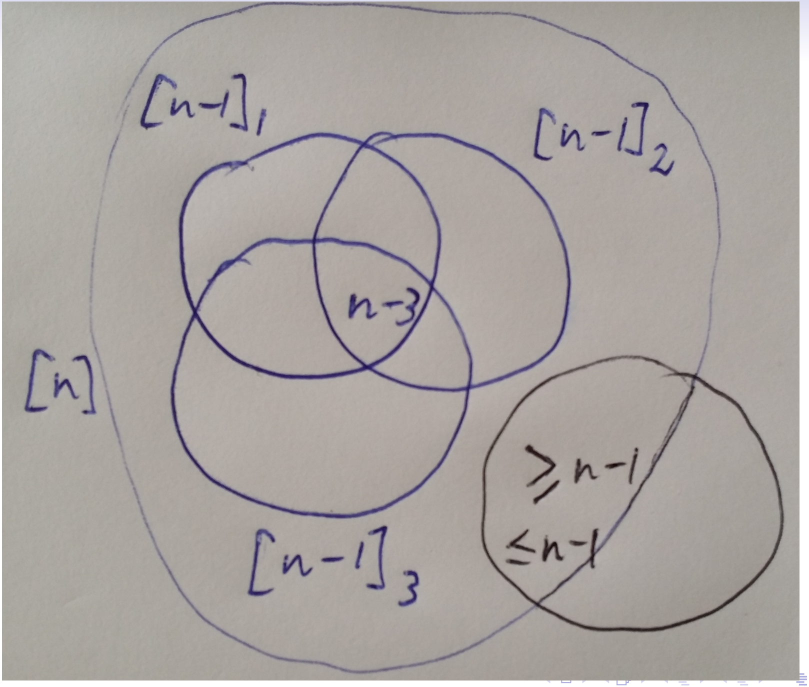
- Remember: An $(n, n-t)$-code C is equivalent to an $(\leq n, n-t)$-code $C' = \{\mathcal{B}(S) \mid S \in C\}$.
- Say *dimension* of $S \in C$ is $\dim(\mathcal{B}(S))$.
- For ≥ 2 codewords, the dimension of each codeword is $n-2$, $n-1$ or n. If a dimension is $n-2$, the code C is a sunflower; so let codeword dimensions be $n-1$ or n.



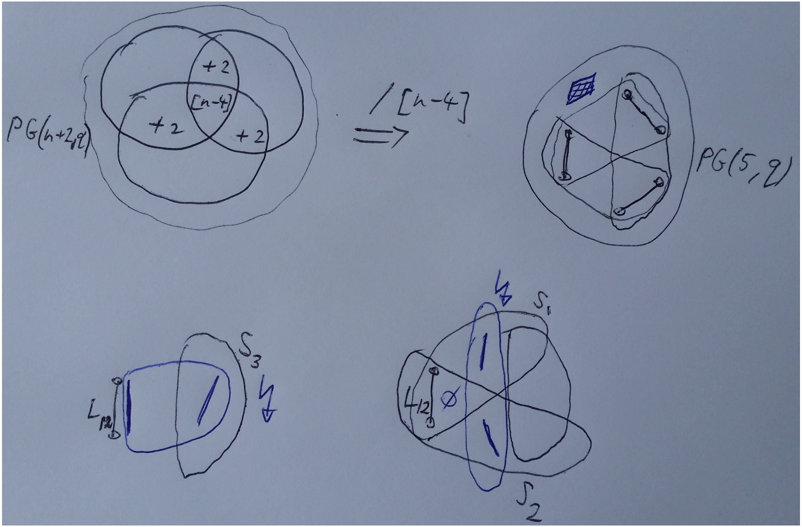


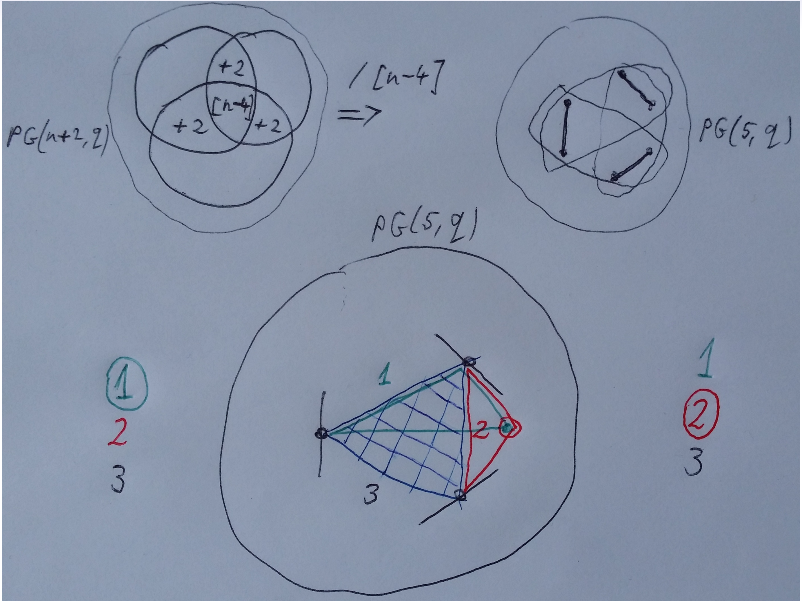


A red codeword intersecting $\langle \overset{I}{0}, \overset{II}{0}, \overset{III}{0} \rangle$ in dimension $n-1$.

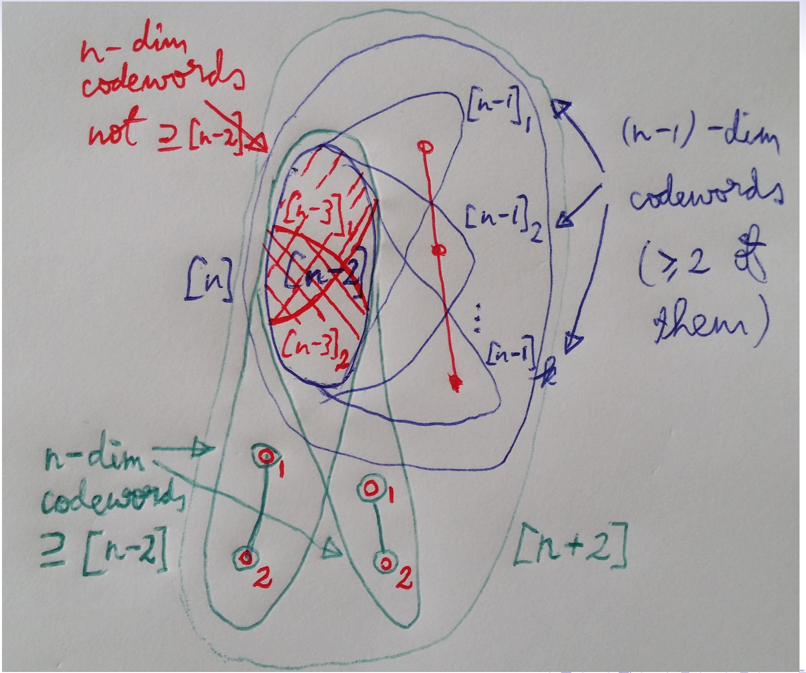


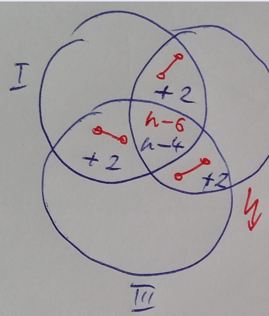
[n-1]$_1$
[n-1]$_2$
n-3
[n]
≥n-1
≤n-1
[n-1]$_3$

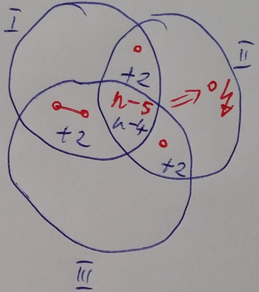

PG(n+2,q)
+2
[n-4]
+2
+2
/[n-4]
PG(5,q)
L12
S3
S1
L12
∅
S2

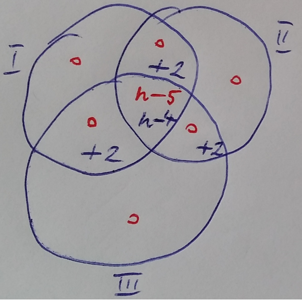


PG(n+2, q)
+2
[n-4]
+2
+2
/ [n-4]
=>
PG(5, q)
PG(5, q)
1
2
3
1
2
3
1
2
3


n-dim codewords not ⊇ [n-2]
[n-1]_1
[n-1]_2
[n-1]_k
(n-1)-dim codewords (≥ 2 of them)
[n-3]_1
[n-2]
[n-3]_2
[n]
n-dim codewords ⊇ [n-2]
[n+2]







A red codeword intersecting $\langle \overset{I}{0}, \overset{II}{0}, \overset{III}{0} \rangle$ in dimension $n-1$.

Thank you!